Towards the security of McEliece's cryptosystem based on Hermitian subfield subcodes

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21st Central European Conference on Cryptology
Debrecen, June 25, 2021

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Introduction

- Recently, there has been a big amount of research addressed to **quantum computers.**
- Quantum computers use quantum mechanical techniques to solve **hard computational problems** in mathematics.
- The existence of these powerful machines threaten many of the **public-key cryptosystem** that are widely in use.
- Robert McEliece (1978) introduced the first **code-based public-key cryptosystem,** that still resists the quantum attack.

Error correcting codes

$[n, k]_q$-code C

- n, k positive integers
- q prime power
- $\mathbb{F}_q$ finite field of order q
- $C \leq \mathbb{F}_q^n$ linear subspace
- $k = \dim(C)$

The code C is given by

- the $k \times n$ **generator matrix** G,
- the $(n-k) \times n$ **parity check matrix** H

$$C = \mathrm{Im} G = \ker H^T$$

Decoding up to t errors

- t positive integer
- $\mathscr{D}_{C,t} : \mathbb{F}_q^n \to C \cup \{*\}$
- $\mathscr{D}_{C,t}(\boldsymbol{y}) = \boldsymbol{x}$, if there is a unique vector $\boldsymbol{e}$ with $\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{e}$ and $\mathrm{wt}(\boldsymbol{e}) \leq t$
- $\mathscr{D}_{C,t}(\boldsymbol{y}) = *$ otherwise

Theorem (Berlekamp, McEliece, van Tilborg 1978)

Given $C, \boldsymbol{y}, t$. It is NP-complete to decide if $\mathscr{D}_{C,t}(\boldsymbol{y}) = *$. (Even for $q = 2$.)

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Code-based cryptography

- Let C be an $[n, k]_q$-code
- given by a generator matrix $G = [I_k G_0]$ in *systematic form*
- Let $\mathscr{D}_{C,t}$ be a decoding up to t errors
- given by **an efficient algorithm**

Example: classic McEliece

- $C_1 = \Gamma(L, g)$ the binary Goppa code with generator matrix G_1.
- S invertible $k \times k$ matrix, P $n \times n$ permutation matrix
- $G = SG_1P$, $C = \mathrm{Im}(G)$
- $\mathscr{D}_{C,t}$ is deduced from S, P and $\mathscr{D}_{C_1,t}$.
- public key: G, t
- private key: $\mathscr{D}_{C,t}$

Encryption:

- plain text message $\boldsymbol{x} \in \mathbb{F}_q^k$
- $\boldsymbol{e} \in \mathbb{F}_q^n$ random with $\mathrm{wt}(\boldsymbol{e}) = t$
- cipher text $\boldsymbol{y} = \boldsymbol{x}G + \boldsymbol{e}$

Decryption:

- $\boldsymbol{z} = \mathscr{D}_{C,t}(\boldsymbol{y}) \in \mathrm{Im}G$
- solve the system of linear equations $\boldsymbol{x}G = \boldsymbol{z}$

Remark. The private key of classic McEliece is (L, g, S, P)

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Security of code-based cryptosystems

Message recovery attack:

- **information set decoding** (ISD)
- implementation: Prange (1962)
- time complexity:

$$C_{\text{Prange}} = \frac{\binom{n}{t}}{\binom{n-k}{t}} \cdot poly(n,k)$$

- Many improvements but still exponential in t

Key recovery attack:

- Hard problem

Distinguishing attack:

- The public key G cannot be distinguished from a **random matrix**
- In many cases, it leads to key recovery method

Definition: Security level

We say that a code-based cryptosystem has security level L (in bits), if

$$\binom{n}{t} \Big/ \binom{n-k}{t} > 2^L.$$

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Schur product distinguisher

- Given two elements $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ in $\mathbb{F}_q^n$, the **Schur product** is the component-wise product
$$\boldsymbol{a} * \boldsymbol{b} := (a_1 b_1, \cdots, a_n b_n)$$
on $\mathbb{F}_q^n$.
- For two linear subspaces $A, B \subseteq \mathbb{F}_q^n$, their **Schur product** is the linear subspace
$$A * B := \mathrm{Span}_{\mathbb{F}_q}\{a * b \mid a \in A \text{ and } b \in B\}.$$
- $A * A$ is denoted as A^{*2}, and we define A^{*k} by *induction.*

Cascudo, Cramer, Mirandola, Zémor (2015)

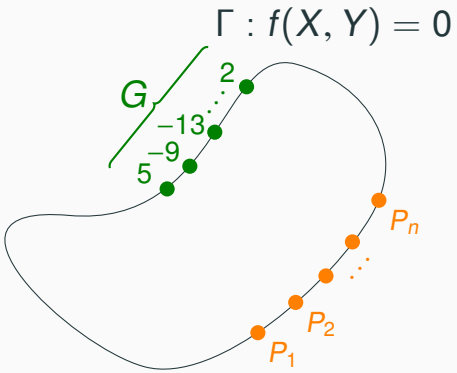
With high probability,
$$\dim(C * C) = \frac{k(k+1)}{2}.$$

Definition: s-good codes

C is **s-good,** if
$$\dim(C * C) = \dim(C^\perp * C^\perp) = n$$

Algebraic-geometric (AG) codes

- Let $\Gamma : f(X, Y) = 0$ be a smooth curve over the finite field $\mathbb{F}_q$.
- Let $P_1, \ldots, P_n$ be distinct places of degree one of Γ.
- Define the divisor $D = P_1 + \cdots + P_n$.
- Let G be a divisor of Γ whose support is disjoint to $\{P_1, \ldots, P_n\}$.
- Let $\mathscr{L}(G)$ be the **Riemann-Roch space** of G.



Definition: AG-code

The **functional code** associated to the divisors D and G is

$$C_L(D, G) = \{(\varphi(P_1), \ldots, \varphi(P_n)) \mid \varphi \in \mathscr{L}(G)\} \leq \mathbb{F}_q^n.$$

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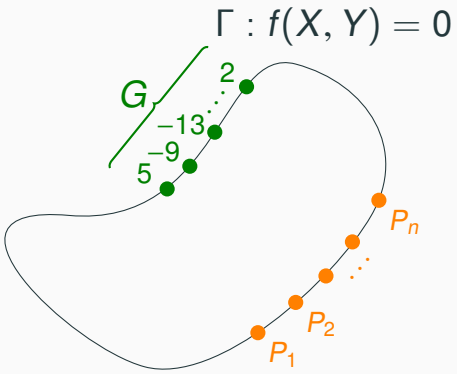

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Fundamental results of AG-codes

Theorem

- *The **length** of $C_L(D, G)$ is n.*
- *For $\deg(G) \leq n$, we have*

$$\dim C_L(D, G) \leq \deg(G) - g + 1,$$

and equality hold for $\deg(G) \geq 2g - 2$.

- *There are **efficient decoding algorithms** for $C_L(D, G)$ up to t errors, where*

$$2t \geq n - \deg(G).$$

Theorem ("Definition" of differential codes $C_\Omega(D, G)$)

*The **dual space** of $C_L(D, G)$ is a code equivalent to $C_L(D, G')$, where $\deg(G') = n + 2g - 2 - \deg(G)$.*

Subfield subcodes of AG-codes

- **Binary Goppa codes, BCH-codes** are subfield subcodes of the **generalized Reed-Solomon codes.**
- Let $r = q^m$, that is, $\mathbb{F}_r$ is an **extension** of $\mathbb{F}_q$ of degree m.
- Let C be an $[n, k, t]_r$ linear code.
- The **subfield subcode** of C is defined by

$$C|_{\mathbb{F}_q} = C \cap \mathbb{F}_q^n.$$

- For $k^* = \dim(C|_{\mathbb{F}_q})$, we have

$$k^* \geq n - m(n-k).$$

- The true values of k^* are not known in general.

Schur filtration of AG-codes (Wieschebrink, Couvreur et al. 2010-2019)

- In general, **AG-codes are s-bad:**

$$\dim(C^{*2}) \leq 2k + g - 1 \ll k(k+1)/2,$$

  where $C = C_L(D, G)$ is an AG-code of dimension k.
- The **Schur product distinguisher** is very effective for AG-codes.
- Using the system of subspaces

$$W_{i,j} = \{z \in \mathbb{F}_q^n \mid z * C^{*i} \leq C^{*j}\}$$

  one can construct an **error correcting array** and an efficient decoding.
- The Schur filtration attack can be extended to subfield subcodes of AG-codes provided the genus g and the degree m are small.

Hermitian subfield subcodes

Definition: Hermitian curve

The equation

$$X^{q+1} = Y^q + Y$$

defines the **Hermitian curve** $\mathscr{H}_q$ over the finite field $\mathbb{F}_{q^2}$.

- Smooth of **genus** $g = q(q-1)/2$.
- q^3 affine points over $\mathbb{F}_{q^2}$.
- Point at infinity: $P_\infty = (0:1:0)$.
- **Maximal curve:** The number of $\mathbb{F}_{q^2}$-rational points attains the Hasse-Weil bound.

1-point Hermitian subfield subcodes

Let $n = q^3$, $r = q^2$, $D = P_1 + \cdots + P_n$. For $s = 1, \ldots, n + 2g - 2$, define

$$C_q(s) = C_L(D, sP_\infty)|_{\mathbb{F}_q}.$$

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S-goodness of 1-point Hermitian subfield subcodes

Proposition

There are positive integers a_q, b_q such that $C_q(s)$ is **s-good** if and only if $a_q \leq s \leq b_q$.

Numerical results using our GAP package HERmitian:

q	4	5	7	8	9	11	13	16
a_q	45	72	192	315	400	720	1 176	2 295
b_q	59	119	335	503	719	1 319	2 183	4 079

Conjecture

$b_q = q^3 - q - 1$.

Comparison of key sizes $k(n-k)\log_2 q$

Classic McEliece	n	k	t	Prange complexity	Key size (bit)
Category 1 (AES-128)	3 488	2 720	64	142.78	2 088 960
Category 3 (AES-192)	4 608	3 360	96	184.89	4 193 280
Category 5 (AES-256)	6 688	5 024	128	262.35	8 359 936
	6 960	5 413	119	263.44	8 373 911
	8 192	6 528	128	300.14	10 862 592

	Code Type	n	k	t	Prange complexity	Key size (bit)
Category 1	$C_{11}(1\,174)$	1 331	927	78	142.33	1 123 524
Category 3	$C_{13}(2\,039)$	2 197	1 735	79	185.89	3 206 280
	$C_{16}(3\,980)$	4 096	3 634	58	187.40	6 715 632
Category 5	$C_{13}(1\,861)$	2 197	1 398	168	263.01	4 468 008
	$C_{16}(3\,874)$	4 096	3 422	111	300.65	9 225 712

THANK YOU FOR YOUR ATTENTION!

Acknowledgement

The presented work was carried out within the project **"Security Enhancing Technologies for the Internet of Things"** 2018-1.2.1-NKP-2018-00004, supported by the National Research, Development and Innovation Fund of Hungary.