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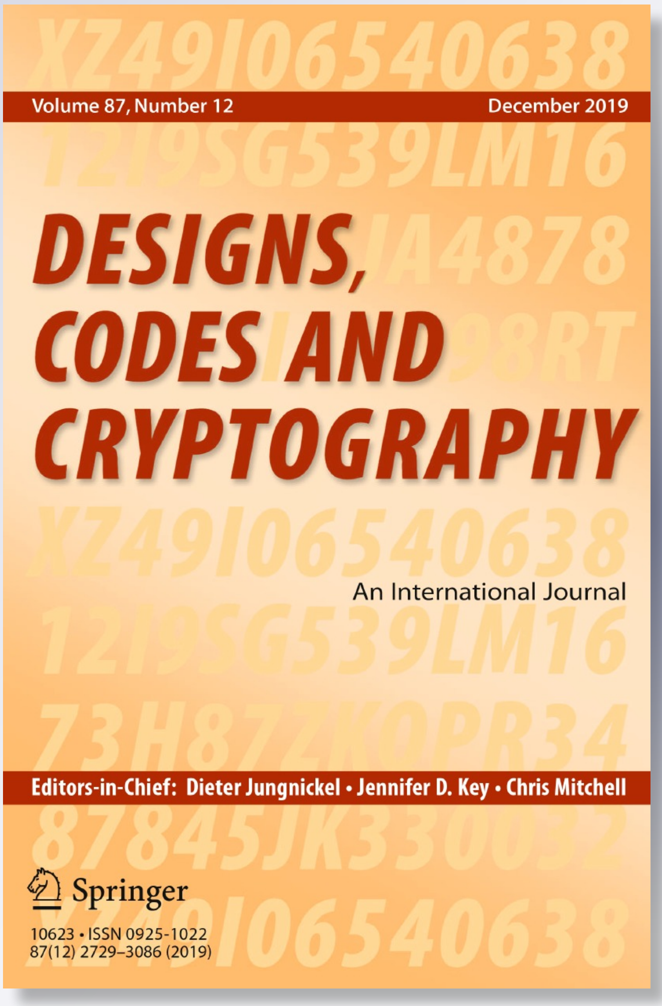

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On the geometry of full points of abstract unitals

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Abstract
The concept of full points of abstract unitals has been introduced by Korchmáros, Siciliano and Szőnyi as a tool for the study of projective embeddings of abstract unitals. In this paper we give a detailed description of the combinatorial and geometric structure of the sets of full points in abstract unitals of finite order.

Keywords Abstract unital · Projective embedding · Perspectivity · Affinity · Full point

Mathematics Subject Classification 51E20 · 05B25

1 Introduction

An abstract unital of order n is a 2-$(n^3+1, n+1, 1)$ design. We say that an abstract unital (X, B) is *embedded* in a projective plane Π if X consists of points of Π and each block $b \in B$ has the form $X \cap \ell$ for some line ℓ of Π. For results on embeddings of abstract unitals see [12] and the references therein.

Let $U = (X, B)$ be an abstract unital of order n and fix two blocks b_1, b_2. Using the terminology of [12], we say that $P \in X$ is a *full point with respect to* (b_1, b_2) if $P \notin b_1 \cup b_2$ and for each $Q \in b_1$, the block connecting P and Q intersects b_2. In other words, there is a well defined projection π_{b_1,P,b_2} from b_1 to b_2 with center P. We denote by $F_U(b_1, b_2)$ the set of full points of U w.r.t. the blocks b_1, b_2. Clearly, $F_U(b_1, b_2) = F_U(b_2, b_1)$.

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The structure of the paper is as follows. In Sect. 2, we give definitions and basic combinatorial properties of full points and related concepts. The main result of this paper is proved in Sect. 3. It shows that for any abstract unital of order q, which is embedded in the Galois plane $\mathrm{PG}(2, q^2)$, the set of full points of two disjoint blocks is contained in a line. Moreover, the perspectivities of two disjoint blocks generate a semi-regular cyclic permutation group acting on each block. In Sect. 4, we give a complete description of the structure of full points in the classical Hermitian unitals. Section 5 gives an overview of computational results about full points in abstract unitals of order 3 and 4, which belong to known classes [1,4,6,13,14]. For the computation we developed and used the GAP package `UnitalSZ` [16].

2 Combinatorial properties of the set of full points

2.1 Bounds on the number of full points

We start with an easy observation on the number of full points of two blocks b_1, b_2 of U. The result seems to be rather weak.

Lemma 2.1 *Let $U = (X, B)$ be an abstract unital of order $n \geq 2$. Then*

$$|F_U(b_1, b_2)| \leq \begin{cases} n^2 - n & \text{if } b_1, b_2 \text{ have a point in common,} \\ n^2 - 1 & \text{if } b_1, b_2 \text{ are disjoint.} \end{cases}$$

Proof For a fixed point $P \in b_1$ we define the set S'_P as the union of the blocks connecting P with $Q \in b_2 \setminus b_1$, and the set $S_P = S'_P \setminus (b_1 \cup b_2)$. Clearly,

$$|S_P| = \begin{cases} n^2 - n & \text{if } b_1, b_2 \text{ have a point in common,} \\ n^2 - 1 & \text{if } b_1, b_2 \text{ are disjoint.} \end{cases}$$

As $F_U(b_1, b_2) \subseteq S_P$, the lemma follows. □

In most (but not all) known examples of abstract unitals, the set of full points is contained in a block. This motivates the following definition.

Definition 2.2 Let $U = (X, B)$ be an abstract unital and $b_1, b_2 \in B$ disjoint blocks.

(i) The triple (U, b_1, b_2) is *full point regular* if the set of full points $F_U(b_1, b_2) \subseteq c$ for some block $c \in B$ such that $b_1 \cap c = b_2 \cap c = \emptyset$.
(ii) The abstract unital U is *full point regular* if for any two disjoint blocks b_1, b_2 the triple (U, b_1, b_2) is full point regular.

2.2 Full points and perspectivities

By definition, any full point P of the blocks b_1, b_2 defines a bijective map $\pi_{b_1,P,b_2} : b_1 \to b_2$; we call it the *perspectivity with center* P.

Definition 2.3 Let b_1, b_2 be blocks of the abstract unital U. Define the *group of perspectivities of* b_1 as

$$\mathrm{Persp}_{b_2}(b_1) = \langle \pi_{b_1,P,b_2} \pi_{b_2,Q,b_1} \mid P, Q \in F_U(b_1, b_2) \rangle.$$

It is easy to see that $\mathrm{Persp}_{b_2}(b_1)$ and $\mathrm{Persp}_{b_1}(b_2)$ are isomorphic permutation groups, the former acting on b_1 and the latter acting on b_2. For different full points Q, R, the perspectivities π_{b_1,Q,b_2} and π_{b_1,R,b_2} are different. This implies $|\,\mathrm{Persp}_{b_2}(b_1)| \geq |F_U(b_1, b_2)|$. In particular, $\mathrm{Persp}_{b_2}(b_1)$ is nontrivial if $|F_U(b_1, b_2)| > 1$. An important case will be when $\mathrm{Persp}_{b_2}(b_1)$ is a cyclic semi-regular permutation group on b_1.

2.3 Dual *k*-nets in abstract unitals

We will present examples of abstract unitals when the set of full points w.r.t. the blocks b_1, b_2 form a third block b_3. More generally, we introduce the concept of an embedded dual k-net of an abstract unital. An abstract k-net is a structure consisting of a set X of points and a set B of blocks, which is partitioned into k disjoint families $B_1, \ldots, B_k$ for which the following hold: (1) every point is incident with exactly one block of every B_i, $(i = 1, \ldots, k)$; (2) two blocks of different families have exactly one point in common; (3) there exist 3 blocks belonging to 3 different B_i which are not incident with the same point. See [3,5] as reference on abstract k-nets.

Definition 2.4 Let $U = (X, B)$ be an abstract unital of order n and $k \geq 3$ an integer. We say that the blocks $b_1, \ldots, b_k$ form an *embedded dual k-net* in U, if the following hold for all $1 \leq i < j \leq k$:

(i) $b_i \cap b_j = \emptyset$.
(ii) For all $P \in b_i$, $Q \in b_j$, the block containing P, Q intersects all $b_1, \ldots, b_k$ in a point.

It is clear that for an embedded dual k-net $b_1, \ldots, b_k$ of U, $b_3 \cup \cdots \cup b_k \subseteq F_U(b_1, b_2)$. The converse needs some explanation.

Lemma 2.5 *Let U be an abstract unital of order n, $k \geq 3$ an integer and $b_1, \ldots, b_k$ blocks of U.*

(i) *If $b_3 \subseteq F_U(b_1, b_2)$, then b_1 and b_2 are disjoint.*
(ii) *If $b_3 \subseteq F_U(b_1, b_2)$, then $b_1 \subseteq F_U(b_2, b_3)$ and $b_2 \subseteq F_U(b_1, b_3)$.*
(iii) *If $b_3 \cup b_4 \subseteq F_U(b_1, b_2)$, then b_3 and b_4 are disjoint.*
(iv) *The blocks $b_1, \ldots, b_k$ form an embedded dual k-net if and only if $b_3 \cup \cdots \cup b_k \subseteq F_U(b_1, b_2)$.*

Proof (i) Assume that $\{Z\} = b_1 \cap b_2$ and $b_3 \subseteq F_U(b_1, b_2)$. Clearly, b_3 is disjoint from $b_1 \cup b_2$. Fix an arbitrary point $P \in b_1 \setminus \{Z\}$. Each point $R \in b_3$ projects P to $b_2 \setminus \{Z\}$. Hence, there are points $R_1, R_2 \in b_3$ such that $\pi_{b_1,R_1,b_2}(P) = \pi_{b_1,R_2,b_2}(P)$. This means that $P \in b_2$, hence $b_1 = b_2$, a contradiction. (ii) For any $P_1 \in b_1$, $P_3 \in b_3$, the block P_1P_3 intesects b_2. Now fix P_1 and let P_3 run through b_3 in order to obtain the bijection π_{b_3,P_1,b_2}. Thus, $P_1 \in F_U(b_2, b_3)$. Since this holds for all $P_1 \in b_1$, the claim follows. For (iii) it suffices to show $b_1 \subseteq F_U(b_3, b_4)$. Take $P \in b_1$, $Q \in b_3$ arbitrary points. From Q, P projects to $R \in b_2$ and using $b_2 \subseteq F_U(b_1, b_4)$, P projects to $S \in b_4$ from R. Hence, Q projects to b_4 from P.

The "only if" part of (iv) follows from the definition. Assume now $b_3 \cup \cdots \cup b_k \subseteq F_U(b_1, b_2)$. By (i) and (iii), all blocks $b_1, \ldots, b_k$ are disjoint. For the indices $3 \leq i < j \leq k$, there is an injective map $\alpha : b_1 \times b_2 \to b_i \times b_j$ mapping $(P_1, P_2) \mapsto (P_i, P_j)$ with collinear quadruple P_1, P_2, P_i, P_j. Moreover α is bijective, hence any pair of points $(P_i, P_j) \in b_i \times b_j$ determines a block b' of U such that $b' \cap b_i = P_i$, $i = 1, 2$. The block joining P_1 and P_2 intersects any block $b_s \subseteq F_U(b_1, b_2)$ in P_s for $3 \leq s \leq k$, therefore $b_1, \ldots, b_k$ form a dual k-net in U. □

2.4 Bounds on dual *k*-nets in abstract unitals

For embedded dual k-nets, the trivial bound is $k \leq n+1$. With some elementary counting, we can improve this to $k \leq n-1$. This implies that an abstract unital of order 3 has no embedded dual 3-nets.

Proposition 2.6 *Let U be an abstract unital of order $n \geq 3$.*

(i) *If U has an embedded dual k-net $\{b_1, \dots, b_k\}$, then $k \leq n-1$.*
(ii) *For two blocks b_1, b_2, $F_U(b_1, b_2)$ cannot contain more than $n-3$ blocks.*

Proof (i) Let us assume that $k > n-1$ and let $\mathcal{P} = b_1 \cup b_2 \cup \ldots \cup b_k$. Any block of U intersects $\mathcal{P}$ in $0, 1, k$ or $n+1$ points, the latter being the blocks b_i themselves. W.l.o.g. consider the disjoint blocks b_1, b_2. Any pair of points chosen from b_1 and b_2 determines the unique block in B which is a k-secant to $\mathcal{P}$, therefore the number of k-secants is $(n+1)^2$. Then, fix an arbitrary block b_i of the dual k-net and a point P on the block b_i. The number of 1-secant blocks on P is n^2-n-2. Thus the number of 1-secant blocks to $\mathcal{P}$ is $k(n+1)\left(n^2-n-2\right)$. Since $|B| = n^2\left(n^2-n+1\right)$ we have

$$k + (n+1)^2 + k(n+1)\left(n^2-n-2\right) \leq n^2\left(n^2-n+1\right),$$

which gives $n^3 - 3n^2 + n + 1 \leq 0$ by $k \geq n \geq 3$, a contradiction.

(ii) If $F_U(b_1, b_2)$ contains the $k-2$ blocks $b_3, \dots, b_k$, then $\{b_1, \dots, b_k\}$ is an embedded dual k-net in U by Lemma 2.5(iv). Hence, $k-2 \leq n-3$ by (i).

□

2.5 Embedded dual 3-nets and latin squares

An embedded dual 3-net $\{b_1, b_2, b_3\}$ determines a latin square L of order $n+1$ in the following way. Label the points of b_1, b_2, b_3 by the set $\{1, \dots, n+1\}$:

$$b_s = \{P_{s,1}, \dots, P_{s,n+1}\}, \qquad s = 1, 2, 3.$$

For $i, j \in \{1, \dots, n+1\}$, let c be the block connecting $P_{1,i}$ and $P_{2,j}$. Define s by $\{P_{3,s}\} = b_3 \cap c$ and write s in row i and column j of L. Choosing a different labeling for b_1, b_2, b_3 results in an *isotope* latin square. By reordering the three blocks, one gets *conjugate* or *parastrophe* latin squares. The set of all parastrophes of a latin square L is also called the *main class* of L. Latin squares are naturally related to (the multiplication tables of) finite *quasigroups*. See [9, Sect. 1.4] for more details and further references on conjugacy and parastrophy of latin squares. On the embedding of latin squares and finite 3-nets in projective planes we refer to [11].

A property which, for each class C, either holds for all members of C or for no member of C is said to be a *class invariant*. Properties of the underlying (dual) 3-nets are *main class invariants* of the corresponding coordinate latin square. In particular, the groups of perspectivities can be defined for (dual) 3-nets and they are useful examples of *main class invariants*. In the primal setting, perspectivities of 3-nets have been presented in [3] and [5].

Let L be a latin square of order n. We say that L is group-based if it is a parastrophe to the Cayley table of a group G of order n. As the group G only depends on the main class of L, the following concept is well-defined.

Definition 2.7 Let $\mathcal{B} = \{b_1, b_2, b_3\}$ be an embedded dual 3-net of the abstract unital U. We say that $\mathcal{B}$ is cyclic, if the corresponding latin square is a parastrophe of the Cayley table of the cyclic group of order $n + 1$, where n is the order of U.

Proposition 2.8 *Let U be an abstract unital of order n and $\mathcal{B} = \{b_1, b_2, b_3\}$ be an embedded dual 3-net of U. The following are equivalent:*

(i) *$\mathcal{B}$ is cyclic.*
(ii) $\mathrm{Persp}_{b_i}(b_j)$ *is the cyclic group of order $n + 1$ for all $1 \le i, j \le 3$, $i \ne j$.*

Proof Let L be the latin square associated to $\mathcal{B}$. By [3, Proposition 1.2], (ii) implies that the rows of L are elements of the cyclic group of order n, hence L is cyclic and (i) holds. Conversely, assume that $\mathcal{B}$ is labeled in such a way that the the coordinate latin square L is the Cayley table of the cyclic group. Then [3, Theorem 6.1] implies (ii).

□

3 Full point regularity of embedded unitals

The questions on the embeddings of abstract unitals in projective planes are long studied problems, with special focus on the embeddings of abstract unitals of order q in the desarguesian plane $\mathsf{PG}(2, q^2)$. Korchmáros et al. [12] introduced the concept of *full point* to study the embedding problem. Their approach was to look at the group of perspectivities with respect to blocks. We notice that while the permutation group $\mathrm{Persp}_{b_2}(b_1)$ depends only on the abstract unital structure of $U = (X, B)$, we may be able compute it more easily when a projective embedding of U is given.

Although the definition of the group of perspectivities works for intersecting blocks b_1, b_2, in the sequel, we will only deal with the case when b_1, b_2 are disjoint. The next definition gives a stronger version of the full point regular property, using the structure of the group of perspectivities.

Definition 3.1 Let $U = (X, B)$ be an abstract unital and $b_1, b_2 \in B$ disjoint blocks.

(i) If (U, b_1, b_2) is a full point regular triple and $\mathrm{Persp}_{b_2}(b_1)$ is a cyclic semi-regular permutation group of b_1, then (U, b_1, b_2) is said to be *strongly full point regular.*
(ii) The abstract unital U is *strongly full point regular* if for any two disjoint blocks b_1, b_2 the triple (U, b_1, b_2) is strongly full point regular.

Notice that U is strongly full point regular if it has no full points at all. The next two lemmas deal with elementary properties of the groups of affinities of projective lines in $\mathsf{PG}(2, q^2)$, where q is a power of the prime p.

Lemma 3.2 *Let p be a prime.*

(i) *Let g be an element of the affine linear group $\mathsf{AGL}(1, p^f)$ such that $o(g) \mid p^f - 1$. Then g has a unique fixed point $v \in \mathbb{F}_{p^f}$ and permutes $\mathbb{F}_{p^f}$ in orbits of length $o(g)$.*
(ii) *Let S be a subgroup of $\mathsf{AGL}(1, p^f)$ such that $p \nmid |S|$. Then, S is cyclic and $|S|$ divides $p^f - 1$. Moreover, S has a unique fixed point in $\mathbb{F}_{p^f}$.* □

Lemma 3.3 *Let ℓ_1, ℓ_2 be two lines of $\mathsf{PG}(2, q^2)$ and P, Q be two points off $\ell_1 \cup \ell_2$. Write $Z = \ell_1 \cap \ell_2$ and $V_i = \ell_i \cap PQ$, $i = 1, 2$. The perspectivity $\pi_{\ell_1,P,\ell_2}\pi_{\ell_2,Q,\ell_1}$ fixes Z and V_1 and permutes $\ell_1 \setminus \{Z, V_1\}$ in orbits of equal lengths.*

Proof Elementary. □

Let S be any set of $n+1$ points in the projective plane Π of order n. A *nucleus* of S is a point P such that each line of Π through P intersects S in a unique point. It follows that $P \notin S$. We denote by $\mathcal{N}(S)$ the set of all nuclei of S.

Let $U=(X,B)$ be a unital of order q embedded in $\mathsf{PG}\left(2,q^2\right)$ and let $b_1,b_2 \in B$ be two (not necessarily disjoint) blocks of U. Denote the lines containing the blocks b_1 and b_2 by ℓ_1 and ℓ_2 respectively. Using the notations in [10] let $\mathcal{B}=b_1 \cup (\ell_2 \setminus b_2)$: the set $\mathcal{B}$ consists of q^2+1 non collinear points, it is contained in the union of the lines ℓ_1 and ℓ_2. Note that $Z=\ell_1 \cap \ell_2$ belongs to $\mathcal{B}$. Let $\mathcal{N}(\mathcal{B})$ denote the set of all nuclei of $\mathcal{B}$. Clearly, if P is a full point w.r.t. to the blocks b_1,b_2 then P is a nucleus of $\mathcal{B}$, hence $F_U(b_1,b_2) \subseteq \mathcal{N}(\mathcal{B})$.

The next lemma formulates [10, Propositions 2 and 3] in our setting.

Lemma 3.4 *Let $U=(X,B)$ be a unital of order q embedded in $\mathsf{PG}\left(2,q^2\right)$ and let $b_1,b_2 \in B$ be two blocks of U. Denote the lines containing the blocks b_1 and b_2 by ℓ_1 and ℓ_2 respectively. Write $Z=\ell_1 \cap \ell_2$ and $\mathcal{B}=b_1 \cup (\ell_2 \setminus b_2)$. Define the set $\Gamma_1=\left\{\pi_{\ell_1,P,\ell_2}\pi_{\ell_2,Q,\ell_1} \mid P,Q \in \mathcal{N}(\mathcal{B})\right\}$ where $\mathcal{N}(\mathcal{B})$ denotes the set of all nuclei of $\mathcal{B}$. Then the following hold:*

(i) Γ_1 *leaves* b_1 *invariant.*
(ii) Γ_1 *is a group of affinities of the affine line* $\ell_1 \setminus \{Z\}$. □

Define the integer r by $q^2=p^r$. The order of the group Γ_1 is tp^h, where $p \nmid t$, and Γ_1 is isomorphic to some group $\Gamma=\mathbf{AB}$ of affinities where $\mathbf{B}$ is an additive subgroup of order p^h of $\mathsf{GF}\left(q^2\right)$ and $\mathbf{A}$ is a multiplicative subgroup of order t of $\mathsf{GF}\left(q^2\right)$ such that $t \mid p^{\gcd(r,h)}-1$. Let $m=\left(p^{r-h}-1\right)/t$ and let $\mathbf{B}_1 \cup \mathbf{O}_1 \cup \ldots \cup \mathbf{O}_m$ be the partition of $\ell_1 \setminus \{Z\}$ into Γ_1-orbits. We have by [10, Sect. 2] that $\mathbf{B}_1$ has length p^h and for each $i=1,2,\ldots,m$ the orbit $\mathbf{O}_i$ has length tp^h.

Let $\mathcal{B}_i=\ell_i \cap \mathcal{B}$ for $i=1,2$ and let $\widehat{\mathcal{B}_1}=\mathcal{B}_1 \setminus \{Z\}$, then $\widehat{\mathcal{B}_1}$ is union of Γ_1-orbits. It follows that the size of $\widehat{\mathcal{B}_1}$ must be divisible by p^h, and we must distinguish between two cases:

(1) If the blocks b_1 and b_2 are disjoint, it means $b_1=\mathcal{B}_1=\widehat{\mathcal{B}_1}$, hence $p^h \mid q+1$. It is possible only for $h=0$, thus the group $\mathbf{B}$ is trivial.
(2) Otherwise $b_1 \cap b_2=\{Z\}$, meaning $b_1=\mathcal{B}_1=\widehat{\mathcal{B}_1} \cup \{Z\}$, hence the size of $\widehat{\mathcal{B}_1}$ is q. In this case $q=ap^h+btp^h$, where $b \in \{0,1,\ldots,m\}$ and $a=1$ or 0, depending on whether $\mathbf{B}_1 \subseteq \widehat{\mathcal{B}_1}$ or not. If $a=0$, then $q=btp^h$, and as $p \nmid t$ we have $t=1$, therefore the group $\mathbf{A}$ is trivial.

Lemma 3.5 *Let $U=(X,B)$ be a unital of order q embedded in $\mathsf{PG}\left(2,q^2\right)$ and let $b_1,b_2 \in B$ be two disjoint blocks of U. Denote the lines containing the blocks b_1 and b_2 by ℓ_1 and ℓ_2 respectively. Write $Z=\ell_1 \cap \ell_2$ and $\mathcal{B}=b_1 \cup (\ell_2 \setminus b_2)$. Define the group Γ_1 generated by the perspectivities $\pi_{\ell_1,P,\ell_2}\pi_{\ell_2,Q,\ell_1}$ with $P,Q \in \mathcal{N}(\mathcal{B})$ where $\mathcal{N}(\mathcal{B})$ denotes the set of all nuclei of $\mathcal{B}$. Then the following hold:*

(i) $p \nmid |\Gamma_1|$.
(ii) Γ_1 *is cyclic and* $|\Gamma_1| \mid q^2-1$.
(iii) Γ_1 *has a unique fixed point* $V_1 \notin b_1 \cup \{Z\}$.
(iv) *The set of full points* $F_U(b_1,b_2)$ *is contained in a line* m *through* V_1 *with* $Z \notin m$.

Proof Assume that Γ_1 has an element γ of order p. Since b_1 is Γ_1-invariant, γ has a fixed point in b_1, different from Z as $Z \notin b_1$. However, affinities with two fixed points have order dividing q^2-1. This proves (i).

Together with Lemmas 3.2 and 3.3, (i) implies (ii) and (iii). Notice that Lemma 3.2(i) is needed to show that $V_1 \notin b_1$.

Since **B** is trivial, the set of nuclei $\mathcal{N}(\mathcal{B})$ is contained in a line m such that $Z \notin m$ (cf. [10, p. 67]). In particular $F_U(b_1, b_2)$ is contained in m as $F_U(b_1, b_2) \subseteq \mathcal{N}(\mathcal{B})$. Furthermore, by Lemma 3.3, for any $P, Q \in \mathcal{N}(\mathcal{B})$ the line PQ contains V_1, hence $V_1 \in m$. This proves (iv). □

We can now state and prove the main theorem of this section.

Theorem 3.6 *If the unital U of order q is embedded in PG$(2, q^2)$ then it is strongly full point regular.*

Proof Let us assume that U is embedded in PG$(2, q^2)$. Let b_1, b_2 be two disjoint blocks of U. If $|F_U(b_1, b_2)| \leq 1$ then we have nothing to prove. Otherwise, by Lemma 3.5 $F_U(b_1, b_2)$ is contained in a block c, which is disjoint to b_1 and b_2. Furthermore, $\mathrm{Persp}_{b_2}(b_1)$ is cyclic, its order divides $q^2 - 1$ and b_1 decomposes into orbits of equal lengths. This means that (U, b_1, b_2) is a strongly full point regular triple. □

4 Full points of the Hermitian unital

For a prime power q, let ρ be a Hermitian polarity of PG$(2, q^2)$. Two points P, Q are said to be *conjugate* if $P \in Q^\rho$. Similarly, the lines ℓ, m are *conjugate* if $\ell^\rho \in m$. Let R^+ be the set of pairs (ℓ, m), where ℓ, m are conjugate lines to each other but not self-conjugate. The projective unitary group PGU$(3, q)$ acts transitively on R^+. Given two conjugate lines ℓ_1, ℓ_2, one constructs $\ell_3 = (\ell_1 \cap \ell_2)^\rho$, conjugate to both ℓ_1 and ℓ_2. We say that ℓ_1, ℓ_2, ℓ_3 form a *polar triangle.* The projective unitary group PGU$(3, q)$ acts transitively on the set of polar triangles. Consider the set X of self-conjugate points of ρ; $|X| = q^3 + 1$. The line ℓ intersects X in 1 or $q + 1$ points, depending on if ℓ is self-conjugate or not. Let ℓ be a non self-conjugate line and m be a line connecting ℓ^ρ and a point $P \in X \cap \ell$. Since $\ell^\rho \in P^\rho$, we have $m = P^\rho$ which must be a self-conjugate line. This means that $(\ell, \ell') \in R^+$ implies that $\ell \cap \ell' \notin X$. It follows that any non self-conjugate line ℓ is contained in exactly $q(q-1)/2$ polar triangles. For further details and background, see [8, Sect. 7.3]

The abstract Hermitian unital $\mathcal{H}(q)$ is constructed from the set X of self-conjugate points of ρ. The subsets cut out by the $(q + 1)$-secants (not self-conjugate lines) form the set B of blocks of $\mathcal{H}(q)$. Notice that we consider $\mathcal{H}(q)$ as an abstract unital, having a natural embedding in PG$(2, q^2)$. The following proposition gives a characterization of the conjugate relation in terms of the abstract unital $\mathcal{H}(q)$ for q even.

Proposition 4.1 *Let q be even, let ρ be a Hermitian polarity of PG$(2, q^2)$ and let X be the set of self-conjugate points of ρ. Let ℓ_1, ℓ_2 be not self-conjugate lines and define the blocks $b_i = \ell_i \cap X$ of $\mathcal{H}(q)$, $i = 1, 2$. Then the following hold:*

(i) *If ℓ_1, ℓ_2 are conjugate, then $F_{\mathcal{H}(q)}(b_1, b_2) = b_3$, where $b_3 = \ell_3 \cap X$ with $\ell_3 = (\ell_1 \cap \ell_2)^\rho$. In other words, the blocks contained in a polar triangle form an embedded dual 3-net of $\mathcal{H}(q)$.*

(ii) *If ℓ_1, ℓ_2 are not conjugate then either $b_1 \cap b_2 \neq \emptyset$, or $|F_{\mathcal{H}(q)}(b_1, b_2)| = 1$.*

Proof (i) Up to projective equivalence, we can assume that the matrix of ρ is the identity. Since the unitary group PGU$(3, q)$ acts transitively on R^+, we can assume $\ell_1 : X_1 = 0$

and $\ell_2 : X_2 = 0$. Then, $\ell_1 \cap \ell_2 = (0, 0, 1)$ and $\ell_3 : X_3 = 0$. Let ε be a $(q+1)$th root of unity in $\mathbb{F}_{q^2}$. The elements of $b_s = \ell_s \cap X$, $s = 1, 2, 3$, have the form

$$A_i = (0, 1, \varepsilon^i), \qquad B_j = (\varepsilon^j, 0, 1), \qquad C_k = (1, \varepsilon^k, 0),$$

respectively, with $i, j, k = 0, 1, \ldots, q$. Since the points A_i, B_j, C_k are collinear if and only if $\varepsilon^{i+j+k} = 1$, we see that A_i projects from C_k to B_{-i-k}. In particular, $b_3 \subseteq F_{\mathcal{H}(q)}(b_1, b_2)$, and equality holds by Theorem 3.6.

(ii) The case when ℓ_1, ℓ_2 are not conjugate and $b_1 \cap b_2 = \emptyset$ was elaborated in [12, Sect. 2.2]. □

Remark 4.2 Proposition 4.1 shows that for q even, $\mathcal{H}(q)$ has embedded dual 3-nets. More precisely, any block of $\mathcal{H}(q)$ is contained in $q(q-1)/2$ polar triangles. The group of automorphisms of $\mathcal{H}(q)$ acts transitively on the set of embedded dual 3-nets.

Let ρ_0 be a Hermitian polarity of the projective line $\mathsf{PG}(1, q^2)$. The set of self-conjugate points of ρ_0 forms a subline $\mathsf{PG}(1, q)$, cf. [8, Lemma 6.2]. Let ℓ be a line of $\mathsf{PG}(2, q^2)$. A *Baer subline* of ℓ is subset of size $q + 1$, consisting of self-conjugate points of some Hermitian polarity ρ of $\mathsf{PG}(2, q^2)$. Equivalently, a Baer subline S is isomorphic to $\mathsf{PG}(1, q)$, and $S = \ell \cap \Pi$ for some line ℓ and a Baer subplane Π.

Proposition 4.3 *Let $U = (X, B)$ be an abstract unital of order q, embedded in $\mathsf{PG}(2, q^2)$. Let b_1, b_2, b_3 form an embedded dual 3-net. Then b_1, b_2, b_3 are Baer sublines.*

Proof Let ℓ be the projective line containing b_1. By Theorem 3.6, $C = \mathrm{Persp}_{b_2}(b_1)$ is a cyclic subgroup of order $q + 1$, preserving b_1. Since C is obtained using projections in $\mathsf{PG}(2, q^2)$, it is a subgroup of the projectivity group of ℓ. By the arguments of [12, Sect. 3] one shows that b_1 is a Baer subline of ℓ. □

Remark 4.4 Let q be even, and consider an *arbitrary* embedding of the Hermitian unital $\mathcal{H}(q)$ in $\mathsf{PG}(2, q^2)$. By Remark 4.2 and Proposition 4.3, all blocks correspond to Baer sublines of $\mathsf{PG}(2, q^2)$. Using the characterization of Hermitian curves from [7,15], plus the arguments of [12, Sect. 3], this observation gives an alternative proof of the uniqueness result of [12] in the even q case.

5 Full points and dual 3-nets of known small unitals

In this section we present computational results on the structure of full points of known small unitals. More precisely, we study the following classes of abstract unitals of order at most 6:

Class BBT 909 unitals of order 3 by Betten et al. [6],
Class KRC 4466 unitals of order 3 with nontrivial automorphism groups by Krčadinac [13],
Class KNP 1777 unitals of order 4 by Krčadinac et al. [14],
Class BB two cyclic unitals of order 4 and 6 by Bagchi and Bagchi [1].

Notice that **KRC** contains all abstract unitals of order 3 with a nontrivial automorphism group. As mentioned in [13], 722 of the **BBT** unitals appear in **KRC**. Moreover, the cyclic **BB** unital of order 4 is contained in **KNP**. The **BB** unital of order 6 has no full points, therefore we omit the **BB** class from the tables of this section. We access the libraries of small unitals and carry out the computations using the GAP4 package `UnitalSZ` [16].

Table 1 **BBT** unitals of order 3

Full points	Group of perspectivities	Unitals
2	C_2	477
2	C_3	94
2	C_4	290

Table 2 **KRC** unitals of order 3

Full points	Group of perspectivities	Unitals
2	C_2	1015
2	C_3	379
2	C_4	897
3	S_4	6

Table 3 **KNP** unitals of order 4

Full points	Group of perspectivities	Unitals
2	C_2	93
2	C_4	71
2	C_5	107
2	C_6	5
3	A_5	2
3	$C_2 \times C_2$	1
3	C_4	32
3	C_5	30
3	S_5	3
4	C_5	8
5	C_5	165
6	$C_5 \rtimes C_4$	72
6	D_{10}	53

5.1 The number of full points and the structure of the group of perspectivities

We only consider disjoint pairs of blocks admitting at least two full points as for only one full point the perspectivitiy group is trivial. In Tables 1, 2 and 3 we summarize the existing number of full points, the structure of the group of perspectivities and the number of unitals with such pairs for each library (**BBT**, **KRC**, **KNP**).

5.2 The structure of the full points

The structure of the full points is only interesting when there is at least 3 of them, hence the **BBT** unitals are out of our scope. Even the case of 3 full points is simple: they are either contained in a block or not. As **KRC** unitals admit at most 3 full points, we are only interested in the **KNP** unitals.

The computation in [16] showed that if there are 4 or 5 full points (in the case of disjoint blocks) then either the whole set of full points is contained in a single block, or no three

Table 4 **KNP** unitals with large full point sets

Set	Property	Cardinality
Ω	At least one *large* full point set	206
A	All large full point sets form a block	74
B	All large full point sets are contained in a block	80
$\overline{B}$	Some large full point sets are not contained a block	126
C	No large full point set is contained in a block	1

Table 5 Full point regularity

Library	Unitals	FPR	SFPR
BBT	909	815	815
KRC	4466	4081	4081
KNP	1777	1586	1582

points are collinear. Similarly in the case of 6 full points either 5 of the full points form a block or no 3 of them are collinear. Now by "collinear" we mean that the points form a subset of some block of the unital.

5.3 Unitals with large full point sets

Let us denote by Ω the subset of unitals with at least one *large* full point set, that is, $|F_U(b_1, b_2)| \geq 3$ for a pair (b_1, b_2) of disjoint blocks. We have seen that Ω is the empty set for **BBT** unitals. By Table 2, $|\Omega| = 6$ for **KRC** unitals. Hence, the interesting case is the **KNP** library, where the size of Ω is 206. In Table 4 we present the number of **KNP** unitals with some restrictions on the structure of full points. Clearly $A \subseteq B, C \subseteq \overline{B}$ and $\Omega = B \cup \overline{B}$.

5.4 Full point regularity

In Table 5 one sees how many of the unitals of the different libraries are full point regular (FPR) and strongly full point regular (SFPR). In fact, if a unital is not strongly full point regular then is not embeddable into $\mathsf{PG}(2, q^2)$. Hence 94 **BBT** unitals, 385 **KRC** unitals and 195 **KNP** unitals are definitely not embeddable into $\mathsf{PG}(2, q^2)$. Notice that [2] proves a much stronger result, where the authors show that there are just two orbits of unitals in $\mathsf{PG}(2, 16)$, containing the Hermitian unitals and Buekenhout–Metz unitals, respectively.

5.5 Embedded dual 3-nets

By Proposition 2.6(ii), one can find embedded dual 3-nets only among the **KNP** unitals. The computation shows us that the latin squares constructed from the dual 3-nets are always of cyclic type, namely, any embedded dual 3-net is cyclic in the **KNP** library. However, we constructed a new unital of order 4 with a non-cyclic embedded dual 3-net, cf. Appendix A.

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Appendix A: Unital of order 4 with non-cyclic embedded dual 3-net

```
LoadPackage("UnitalSZ");

bls:=[[1,2,55,64,65], [1,3,32,46,63], [1,4,7,34,45], [1,5,11,31,44],
    [1,6,12,19,54], [1,8,38,47,50], [1,9,24,27,40], [1,10,20,48,53],
    [1,13,17,49,57], [1,14,15,16,29], [1,18,33,43,58], [1,21,23,25,37],
    [1,22,51,56,60], [1,26,30,39,52], [1,28,36,41,62], [1,35,42,59,61],
    [2,3,6,30,58], [2,4,14,54,60], [2,5,29,46,47], [2,7,13,48,59],
    [2,8,34,37,40], [2,9,10,18,31], [2,11,19,32,52], [2,12,20,50,57],
    [2,15,21,43,62], [2,16,23,27,28], [2,17,33,45,61], [2,22,24,25,26],
    [2,35,38,39,41], [2,36,49,53,56], [2,42,44,51,63], [3,4,19,23,33],
    [3,5,10,39,59], [3,7,22,49,52], [3,8,14,48,65], [3,9,25,29,60],
    [3,11,15,20,34], [3,12,13,16,61], [3,17,24,28,44], [3,18,47,53,57],
    [3,21,36,40,42], [3,26,37,38,43], [3,27,35,56,64], [3,31,45,55,62],
    [3,41,50,51,54], [4,5,41,52,53], [4,6,26,31,47], [4,8,16,36,57],
    [4,9,56,58,59], [4,10,28,46,65], [4,11,21,50,64], [4,12,35,44,62],
    [4,13,30,32,51], [4,15,37,39,61], [4,17,18,20,25], [4,22,27,38,48],
    [4,24,42,49,55], [4,29,40,43,63], [5,6,7,32,37], [5,8,42,54,58],
    [5,9,12,17,51], [5,13,27,36,63], [5,14,22,61,62], [5,15,25,40,49],
    [5,16,19,20,26], [5,18,21,28,38], [5,23,30,55,60], [5,24,33,48,64],
    [5,34,35,57,65], [5,43,45,50,56], [6,8,56,62,63], [6,9,21,61,65],
    [6,10,14,40,41], [6,11,25,43,51], [6,13,38,44,55], [6,15,42,46,57],
    [6,16,22,34,64], [6,17,23,36,52], [6,18,48,49,60], [6,20,28,45,59],
    [6,24,35,50,53], [6,27,29,33,39], [7,8,20,24,51], [7,9,41,63,64],
    [7,10,11,42,60], [7,12,15,55,56], [7,14,23,26,35], [7,16,44,46,53],
    [7,17,29,38,62], [7,18,19,39,50], [7,21,27,31,57], [7,25,47,58,65],
    [7,28,30,33,40], [7,36,43,54,61], [8,9,11,13,46], [8,10,12,45,52],
    [8,15,18,27,59], [8,17,21,35,60], [8,19,43,49,64], [8,22,29,30,53],
    [8,23,32,39,44], [8,25,31,33,41], [8,26,28,55,61], [9,14,52,55,57],
    [9,15,19,28,53], [9,16,35,43,47], [9,20,22,36,39], [9,23,48,50,62],
    [9,26,32,33,42], [9,30,34,38,54], [9,37,44,45,49], [10,13,23,34,43],
    [10,15,17,30,64], [10,16,21,32,56], [10,19,25,35,55], [10,22,33,54,57],
    [10,24,36,37,47], [10,26,27,51,62], [10,29,44,50,61], [10,38,49,58,63],
    [11,12,33,38,59], [11,14,39,47,56], [11,16,18,54,62], [11,17,22,41,65],
    [11,23,24,29,57], [11,26,36,45,48], [11,27,30,49,61], [11,28,35,37,63],
    [11,40,53,55,58], [12,14,24,30,43], [12,18,23,42,65], [12,21,26,41,49],
    [12,22,28,32,47], [12,25,34,48,63], [12,27,37,53,60], [12,29,31,36,58],
    [12,39,40,46,64], [13,14,21,33,53], [13,15,41,45,47], [13,18,26,29,64],
    [13,19,24,31,56], [13,20,35,52,58], [13,22,37,42,50], [13,25,28,39,54],
    [13,40,60,62,65], [14,17,19,59,63], [14,18,37,46,51], [14,20,31,38,42],
    [14,25,36,44,64], [14,27,32,45,58], [14,28,34,49,50], [15,22,23,31,63],
    [15,24,32,38,65], [15,26,50,58,60], [15,33,35,36,51], [15,44,48,52,54],
    [16,17,37,48,58], [16,24,41,59,60], [16,25,30,42,45], [16,31,39,49,65],
    [16,33,50,55,63], [16,38,40,51,52], [17,26,34,46,56], [17,27,47,54,55],
    [17,31,32,40,50], [17,39,42,43,53], [18,22,35,40,45], [18,24,52,61,63],
    [18,30,41,44,56], [18,32,34,36,55], [19,21,22,44,58], [19,27,34,41,42],
    [19,29,45,51,65], [19,30,37,57,62], [19,36,38,46,60], [19,40,47,48,61],
    [20,21,30,47,63], [20,23,40,54,56], [20,27,43,44,65], [20,29,37,41,55],
    [20,32,60,61,64], [20,33,46,49,62], [21,24,45,46,54], [21,29,34,52,59],
    [21,39,48,51,55], [22,43,46,55,59], [23,38,45,53,64], [23,41,46,58,61],
    [23,47,49,51,59], [24,34,39,58,62], [25,27,46,50,52], [25,32,53,59,62],
    [25,38,56,57,61], [26,40,44,57,59], [26,53,54,63,65], [28,29,42,48,56],
    [28,31,43,52,60], [28,51,57,58,64], [29,32,35,49,54], [30,31,35,46,48],
    [30,36,50,59,65], [31,34,51,53,61], [31,37,54,59,64], [32,41,43,48,57],
    [33,34,44,47,60], [33,37,52,56,65], [39,45,57,60,63], [42,47,52,62,64]
];;

u:=AbstractUnitalByDesignBlocks(bls);
t:=BlocksOfUnital(u){[1,33,200]};
StructureDescription(PerspectivityGroupOfUnitalsBlocks(u,t[1],t[2],t[3]));
```

The output of the last command is `"S5"`, showing that the group of perspectivities is not cyclic.

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